the order of l_1^+) from the axis of symmetry (y = 0) breaking of a whole fiber occurs.

The results of the above calculations are in qualitative agreement with the conclusions obtained from experiments.

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FREE TORSIONAL OSCILLATIONS OF A STANDARD LINEAR BODY

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One of the problems of the torsional oscillations of a metal relaxing rod is considered in [1]. The behavior of the system in a time t is characterized by a function $\varphi(z, t)$, which defines the angle of rotation around the axis of the rod of an infinitely thin horizontal layer of material. The initial equation for the relaxation time $\tau \to \infty$ reduces to a wave-type equation which describes the motion of an idealized elastic material [2, 3].

However, the solution obtained in [1] as $\tau \to \infty$ is independent of the time, and hence does not agree with the solution of the similar problem for absolutely elastic materials [4]. This is due to the fact that when formulating the initial and boundary conditions in [1], zero initial values of the velocity and acceleration of the motion of the system were assumed for t = 0 over the whole specimen, whereas from the physical point of view motion of the system is only possible if its acceleration is different from zero.

We will consider the free torsional oscillations of a cylindrical uniform isotropic viscoelastic rod of radius R and length $h \gg 2R$, and a connected rigid disk. We will assume that the amplitude of the torsional oscillations of the distributed mass is small, the transverse cross sections S(z) of the rod are not distorted, and are not displaced along the z axis (S(z) = const), and the torsion is not accompanied by a change in the volume of the deformed mass [1]. The z axis of a cylindrical system of coordinates (r, α, z) coincides with the axis of the rod. To determine the initial state of the system we will assume that before starting the pendulum the rod is twisted about the z axis by the continuous torsional moment of a pair of forces P concentrated on the boundary S(z = h). Suppose that during a fairly large instant of time t_0 the rod reaches its initial statically loaded state. Then, for $t \geq t_0$ the torsional moment of the forces (PR_0) will be constant over the whole area of existence of the deformed mass $0 \leq z \leq h$, and is defined in the form

$$PR_0 = \frac{\mu\pi}{2} R^4 \partial\varphi(z)/\partial z, \tag{1}$$

where $\varphi(z)$ is the angle of rotation of the cross sections S(z) (around the z axis) for a statically twisted state of the rod. If when $t_1 \geq t_0$ the forces P are simultaneously and instantaneously removed, the connected disk begins to change into a state of rotational motion around the z axis. We will assume that the relaxation

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time $\tau = \eta/(G_0 - \mu)$, where η is the coefficient of viscosity, G_0 is the instantaneous shear modulus, and μ long-term shear modulus, where $G_0 > \mu$, and may approach infinity both as $\eta \to \infty$ ($G_0 \neq \mu$), and as $G_0 \to \mu$ ($\eta \ll \infty$). The equilibrium of the moments of the forces on the boundary $S(h)$ of the rod as $G_0 \to \mu$ and $t \geq t_1$ is given by the relation

$$\mu\pi R^4\varphi_z(h, t) = -2I\varphi_{tt}(h, t). \tag{2}$$

If when $t < t_0$ the coefficient η is small, and when $t > t_0$ it is large ($\eta \to \infty$, which can be achieved by cooling the twisted specimen), the equilibrium moments of the forces on the boundary $S(z = h)$ can be expressed by the relation ($t \geq t_1$)

$$\begin{gathered}\pi R^4[G_0\varphi_z(h, t) - (G_0 - \mu)\gamma] = -2I\varphi_{tt}(h, t),\\ \gamma = 2PR_0/\mu\pi R^4.\end{gathered} \tag{3}$$

The differential relation between the local relative deformation ε and the stress σ of a plane-parallel shear under isothermal conditions of motion of the system has the form [2, 3]

$$\sigma + \tau\dot{\sigma} = \mu\varepsilon + G_0\tau\dot{\varepsilon}. \tag{4}$$

From relation (4) we obtain the initial equation of motion of the rod [2, 3]

$$\tau\rho\varphi_{ttt} + \rho\varphi_{tt} = \mu\varphi_{zz} + G_0\tau\varphi_{tzz}, \tag{5}$$

where ρ is the density of the material. Equation (41) in [1] is identical with (5) when $G_0\tau = \tau\mu + \eta$. Taking $t_1 = 0$ as the instant when the pendulum is released, the two initial conditions of motion of the system can be written in the form

$$\varphi(z, 0) = \gamma z, \ \varphi_t(z, 0) = 0 \text{ for } 0 \leqslant z \leqslant h. \tag{6}$$

Since in a moving system the energy dissipation processes are delayed with respect to the instant when the torsional moment begins to vary, at the instant $t_1 = 0$ the rod behaves as an idealized elastic system [4, 5]. In this connection, the initial conditions of problems (5) must agree with the boundary conditions of the analogous problem for idealized elastic materials, i.e., with conditions (2) and (3). In our case, it is sufficient to require that the boundary conditions of problem (5) should agree with condition (3), since (2) is a special case of (3).

On the basis of these assumptions, using the law of conservation of energy we will derive the third of the initial conditions. We will determine the reserve of potential energy $E_{y0}(z)$ of elastic deformation of an infinitely thin horizontal layer of material with coordinate z and area $S = \pi R^2$ (i.e., the running energy density of the rod). Using the above assumptions and approximations, the value of $E_{y0}(z)$ can be written in the form

$$E_{y0}(z) = \frac{\mu}{2}\int_0^R\int_0^{2\pi}\varepsilon_0^2 r\,dr\,d\alpha,$$

where $\varepsilon_0 = \partial u(z, r)/\partial z = r\partial\varphi(z)/\partial z$ is the relative shear of an infinitely small element of volume in an infinitely thin layer of material with coordinate z, and $u(z, r) = r\varphi(z)$ is a function which gives the value of the plane-parallel displacement of this element in the direction of action of the moment of the forces $r\sigma(z, r)$ ($r\,dr\,d\alpha = ds$).

The total energy E_t and the shear stress $\sigma(z, r)$ of the rod, taking (1) into account, are given by

$$E_t = \int_0^h E_{y0}(z)\,dz = P^2R_0^2h/\mu\pi R^4,$$

$$\sigma(z, r) = \mu\varepsilon_0 = r\mu\gamma = 2PR_0r/\pi R^4.$$

These results indicate that in a statically twisted rod the potential energy is only conserved in the Hook section of the model employed. At the same time the energy and the stress in the Maxwell section are zero.

We will assume that at the instant t_1 the rod behaves as an elastic system. In this case the relative deformation and the energy of the Hook section decrease from ε_0 and $\mu\varepsilon_0^2/2$ to $\varepsilon = \varepsilon(z, r, t)$ and $\mu\varepsilon^2/2$, while the elastic element of the Maxwell section increases in absolute value from zero to $(\varepsilon - \varepsilon_0)$ and $(G_0 - \mu)(\varepsilon - \varepsilon_0)^2/2$, respectively. Nevertheless a considerable part of the potential energy E_p is converted into kinetic energy when the rotation around the z axis of the distributed mass and the connected disk accelerates.

Taking all these effects into account, on the basis of the law of conservation of energy of the system we have

$$E_t = \frac{1}{2}\left\{I\varphi_t^2(h, t) + \int_0^h\int_0^R\int_0^{2\pi}\left[(G_0-\mu)(\varepsilon-\varepsilon_0)^2 + \mu\varepsilon^2 + \rho u_t^2(z, r, t)\right] r dr d\alpha dz\right\}. \tag{7}$$

It can be shown that the partial derivative with respect to time of (7) (with $\varphi_t(0, t) = 0$) reduces to the form

$$[-(G_0-\mu)\gamma + G_0\varphi_z(h, t) + 2I\varphi_{tt}(h, t)/\pi R^4]\,\varphi_t(h, t) = \int_0^h [G_0\varphi_{zz}(z, t) - \rho\varphi_{tt}(z, t)]\,\varphi_t(z, t)\,dz. \tag{8}$$

Further, differentiating (8) with respect to z, we have

$$\rho\varphi_{tt}(z, t) = G_0\varphi_{zz}(z, t). \tag{9}$$

Substituting (9) into (8), we arrive at (3). Changing to the limit $t \to +0$ in (9) and (8) (taking into account certain properties of (8) and (9)), we obtain the third initial condition of problem (5) in the form

$$\rho\varphi_{tt}(z, 0) = G_0\varphi_{zz}(z, 0) \text{ for } 0 \leqslant z < h; \tag{10}$$

$$\pi R^4[G_0\varphi_z(h, 0) - (G_0-\mu)\gamma] = -2I\varphi_{tt}(h, 0) \text{ for } z = h. \tag{11}$$

Relation (10) is the equation of the distributed mass inside the specimen for $t = 0$ [5], while (11) are the equilibrium moments of the forces on the boundary $S(z = h)$ and close to it.

The boundary conditions of the motion of a viscoelastic system can be written in the form ($t > 0$) [1]

$$\begin{gathered}\pi R^4[\mu\varphi_z(h, t) + G_0\tau\varphi_{tz}(z, t)] = -2I[\tau\varphi_{ttt}(h, t) + \varphi_{tt}(h, t)],\\ \varphi(0, t) = 0,\ \varphi_t(0, t) = 0.\end{gathered} \tag{12}$$

The solution of the problem (5), (6) and (10)–(12) will be found by the method of separation of variables [2–4]

$$\varphi(z, t) = \sum_{k=1}^{\infty}\Psi_k(z)\,T_k(t),$$

where $\Psi_k(z)$ are functions which define the form of the oscillations of the specimen, while $T_k(t)$ are functions which give the nature of the variation of the amplitudes of the oscillations with time. Using (5) we will write the characteristic equation for $T_k(t)$ [3]

$$\xi_k^3 + \xi_k^2/\tau + \beta_k^2(\mu + G_0\tau\xi_k)/\tau\rho h^2 = 0, \tag{13}$$

where β_k are the positive solutions of the equation

$$\operatorname{ctg}\beta = 2I\beta/\rho\pi R^4 h. \tag{14}$$

The roots of Eq. (13) will be represented in the form [1, 5]

$$\xi_k^{(1)} = -\frac{1}{3\tau} + \alpha_k,\ \xi_k^{(2,3)} = -\frac{1}{3\tau} - \frac{\alpha_k}{2} \pm i w_k. \tag{15}$$

The quantities α_k and w_k are found by the well-known method described in [1, 5]. If all three roots of (15) take negative real values, they indicate that the motion of the system is aperiodically limited [3, 5].

The solution of Eq. (5), taking (6)–(15) into account, can be written in the form

$$\varphi(z, t) = \sum_{k=1}^{\infty}\Psi_k(z)\exp\left(-\frac{t}{3\tau}\right)\left[\frac{C_k e^{\alpha_k t} + (M_k\cos w_k t + N_k\sin w_k t)\,e^{-\frac{\alpha_k}{2}t}}{3\alpha_k^2 + p_k}\right], \tag{16}$$

where

$$\Psi_k(z) = \frac{4\gamma h\sin\beta_k\cdot\sin\beta_k\frac{z}{h}}{\beta_k(2\beta_k + \sin 2\beta_k)};$$

$$C_k = \alpha_k^2 + \alpha_k/3\tau - 2/9\tau^2 + \beta_k^2(G_0-\mu)/\rho h^2;$$

$$M_k = 2\alpha_k^2 + \mu\beta_k^2/\rho h^2 - \alpha_k/3\tau - 1/9\tau^2;$$

$$N_k = \frac{1}{w_k}\left[\frac{\alpha_k^2}{2\tau} + \frac{\alpha_k}{6\tau^2} + \frac{p_k}{3\tau} - \frac{\alpha_k\beta_k^2}{2\rho h^2}(2G_0 - 3\mu)\right];$$

$$p_k = G_0\beta_k^2/\rho h^2 - 1/3\tau^2.$$

To determine the conditions under which the results are applicable it is necessary to investigate the behavior of the solution (16) in different experimental situations. A method of carrying out this analysis is given in [5].

If we put $I = 0$ in the solution (16) and take the limit $G_0 \to \mu$, it takes the form of a function which agrees with the solution of the analogous problem for an idealized elastic material [4]. Since in our case $I \neq 0$, the behavior of the solution (16) both as $G_0 \to \mu$ and when $I \neq 0$ is of interest

$$\varphi(z, t) = \sum_{k=1}^{\infty} \Psi_k(z) \cos \beta_k \sqrt{\frac{\mu}{\rho h^2}}\, t,$$

and also as $\eta \to \infty$ and when $I \neq 0$

$$\varphi(z, t) = \frac{(G_0 - \mu)\gamma z}{G_0} + \frac{\mu}{G_0} \sum_{k=1}^{\infty} \Psi_k(z) \cos \beta_k \sqrt{\frac{G_0}{\rho h^2}}\, t.$$

In the first case, the elastic behavior of the rod is characterized by a single shear modulus μ, and in the second case it is characterized by two values of μ and G_0. The presence of two limits of the solution (16) is due to the fact that the relaxation time of a standard linear body can take extremely large values as $G_0 \to \mu$ and $\eta \to \infty$.

We will put $h = 10$ cm, $\rho = 1.6$ g/cm^3, $\eta = 6 \cdot 10^7$ P, $\mu = 3 \cdot 10^7$ dynes/cm^2, $G_0 = 9 \cdot 10^7$ dynes/cm^2, and $\beta_1 \cong \sqrt{\rho \pi h R^4 / 2I} \cong 10^{-2}$. We will substitute these values into the functions $\varphi(z, t)$ obtained by the different methods. As a result of this for the first term of the sum of the series (16) (with index $k = 1$), representing the principal part of $\varphi(z, t)$, with $t = \pi / w_1$, $3\alpha_1^2 + p_1 = d_1$ in $\Psi_1(z)$, $T_1(t) = \varphi_1(z, t)$ we have $C_1/d_1 \cong 2/3$, $M_1/d_1 \cong 1/3$, $T_1(t) \cong 0.284$. At the same time the analogous quantities, but defined from Eq. (60) in [1] (with our method of numbering the roots of the equation of the form (14)), take the following values: $C_0^{(1)}/d_1 \cong 1.002$, $M_0^{(1)}/d_1 \cong 10^{-3}$, $T_0^{(1)}(t) \cong 0.864$, respectively. If we substitute into $\varphi(z, t)$ $\rho = 2.7$ g/cm^3, $\eta = 2.2 \cdot 10^{16}$ P, $G_0 \cong 2.55 \cdot 10^{11}$ dynes/cm^2, $\mu = 2.33 \cdot 10^{11}$ dynes/cm^2, and $\beta_1 \cong 1.29 \cdot 10^{-4}$, then for $\varphi_1(z, t)$ we will have $C_1/d_1 \cong 0.086$, $M_1/d_1 \cong 0.914$, $T_1(t) \cong -0.828$, and also $C_0^{(1)}/d_1 \cong 1$, $M_0^{(1)}/d_1 \cong 1.96 \cdot 10^{-14}$, $T_0^{(1)}(t) \cong 1$. It can be seen that the quantities calculated above in Eq. (16) differ considerably from those of solution (60) in [1]. Note that the coefficient M_1/d_1, representing the principal part of the amplitude of the oscillations of the pendulum in $\varphi(z, t)$, as τ increases in (16) approaches 1 as $G_0 \to \mu$, while in (60) of [1] it decreases to a practically inappreciable value (i.e., $M_0^{(1)}/d_1 \leq 10^{-14}$). These facts suggest that the solution of the type (60) in [1] does not describe the motion of a torsional pendulum either as $\tau \to \infty$, or for real finite values of $\tau \leq 10^{+6}$ sec. The solution (16) does not contradict physical ideas of the nature of the oscillations of a pendulum for any values of $\tau \geq 0$.

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